Chapter 2 in *Convex Optimization of Power Systems*.

1 Basic electrical quantities

Three-phase, balanced alternating current:

\[
\begin{align*}
v_k(t) &= \bar{V} \cos(\omega t + \phi_k) \\
i_k(t) &= \bar{I} \cos(\omega t + \phi_k - \psi) \\
\phi_k &= \frac{2\pi k}{3}, \quad k = 0, 1, 2
\end{align*}
\]

(Remember, \(v_k(t)\) is voltage difference to ground.)

Instantaneous power (in a phase):

\[
p_k(t) = v_k(t)i_k(t) = \frac{\bar{V}\bar{I}}{2} \cos(\psi)(1 + \cos(2(\omega t + \phi_k))) + \frac{\bar{V}\bar{I}}{2} \sin(\psi) \sin(2(\omega t + \phi_k))
\]

Why three phases?

- Zero return current: \(\sum_k i_k(t) = 0\)
- Constant instantaneous power: \(\sum_k p_k(t) = \frac{3\bar{V}\bar{I}\cos(\phi)}{2}\).

Phasors: all parameters are constant (steady-state).

\[
\begin{align*}
V_k &= \frac{\bar{V}}{\sqrt{2}} e^{j\phi_k} \\
I_k &= \frac{\bar{I}}{\sqrt{2}} e^{j(\phi_k - \psi)}
\end{align*}
\]

Phasor power quantities:
• Complex power (in one phase)

\[ S = V_k I_k^* = \frac{\bar{V} \bar{I}}{2} (\cos(\psi) + j \sin(\psi)) \]

• Real power

\[ P = \text{real}[S] = \frac{\bar{V} \bar{I}}{2} \cos(\psi) = \text{mean}[p_k(t)] \]

• Reactive power

\[ Q = \text{imag}[S] = \frac{\bar{V} \bar{I}}{2} \sin(\psi) \]

What are real and reactive power? Observe:

\[ p_k(t) = P(1 + \cos(2(\omega t + \phi_k))) + Q \sin(2(\omega t + \phi_k)) \]

\( P \) is the coefficient of the non-zero mean part. \( Q \) is the coefficient of the zero mean part.

Per phase analysis: symmetry between lines enables us to just analyze one phase, drop \( k \) index.

2 Power flow

Consider a power line with phasor impedance \( Z_{12} \) and admittance \( Y_{12} = g_{12} - j b_{12} \). Ohm’s law:

\[ I_{12} = (V_1 - V_2)Y_{12} \]

(a linear relationship)

The power flow is:

\[ S_{12} = V_1 I_{12}^* = V_1(V_1 - V_2)^*Y_{12}^* \]

(a quadratic relationship)

Why not

\[ L_{12} = (V_1 - V_2)(V_1 - V_2)^*Y_{12}^* = |V_1 - V_2|^2 Y_{12}^*? \]

This is the loss in the line. Observe:

\[ S_{12} + S_{21} = V_1(V_1 - V_2)^*Y_{12}^* + V_2(V_2 - V_1)^*Y_{12}^* \]

\[ = (V_1 - V_2)(V_1 - V_2)^*Y_{12}^* \]

\[ = L_{12} \]
3 Optimization

\( f(x), x \in \mathbb{R}^n \).

- \( x_0 \) is a global minimum of \( f(x) \) if \( f(x_0) \leq f(x) \) for all \( x \).
- \( x_0 \) is a local minimum of \( f(x) \) if \( f(x_0) \leq f(x) \) for all \( x \) s.t. \( \|x - x_0\| \leq \epsilon \), \( \epsilon > 0 \).

Convexity:

- Function: \( f(x) \) is convex if:
  \[
  f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)
  \]
  for all \( 0 \leq \alpha \leq 1 \).
- Set: \( \mathcal{X} \) is convex if \( x, y \in \mathcal{X} \) implies \( \alpha x + (1 - \alpha)y \in \mathcal{X} \).
- If \( g(x) \) is convex, then \( \mathcal{X} = \{x | g(x) \leq 0\} \) is convex.

Optimization problem:

\[
\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t.} \quad g_i(x) \leq 0, \ i = 1, \ldots, m
\]

If \( f(x) \) and all \( g_i(x) \) are convex, then any local minimum is a global minimum ...
... a convex optimization problem.

Computational tractability:

- Convex optimization: easy, often polynomial-time
- Nonconvex optimization: hard, often NP-hard

(NP-hard: no polynomial-time (efficient) algorithm can exist)

3.1 Linear programming

(slight misnomer, affine is more accurate)

- \( f(x) = c^T x \)
• Affine constraints: \( g_i(x) = a_i^T x - b_i \) (usually as vector: \( Ax \leq b \))
• Easiest type of optimization
• Solvable in PT with an IP method, or very fast with simplex method.
• Quadratic programming with \( f(x) = x^T C x \) is also easy if \( C \succeq 0 \) (psd).
• Is it convex? Check definition, yes.

3.2 Mixed-integer programming

\[
\begin{align*}
\min_{x,y} & \quad f(x, y) \\
\text{s.t.} & \quad g_i(x, y) \leq 0, \ i = 1, \ldots, m \\
& \quad y_i \in \mathbb{Z} \text{ (the integers)}
\end{align*}
\]

• NP-hard even when \( f \) and \( g_i \) are all linear!
• Branch-and-bound and cutting planes are powerful heuristics
• \( y_i \in \mathbb{Z} \) is nonconvex!
• As we’ll see, common in power systems.

3.3 Semidefinite programming

Positive semidefinite:
• \( X \in \mathbb{C}^{n \times n}, \) Hermitian: \( X = X^* \) (conjugate transpose)
• Definition: \( z^* X z \geq 0 \) for all \( z \in \mathbb{C}^n \)
• Equivalent: all eigs. of \( X \) nonnegative, all principal minors nonnegative
• Notation: \( X \succeq 0 \)

\( X \succeq 0 \) is convex constraint.
Proof: Suppose \( X, Y \succeq 0 \). Then
\[
z^*(\alpha X + (1 - \alpha) Y)z = \alpha z^* X z + (1 - \alpha) z^* Y z \geq 0.
\]
Done!
A semidefinite program (SDP):

\[
\begin{align*}
\min_X & \quad \text{trace}(CX) \\
\text{s.t.} & \quad \text{trace}(A_iX) = b_i \\
& \quad X \succeq 0
\end{align*}
\]

Features of SDP:

- Convex, 1 minimum
- Generalization of LP (don’t solve LP as SDP)
- SDP’s can be solved in polynomial-time using interior point method.

3.3.1 Example: eigenvalue optimization

Suppose \( A(x) \in \mathbb{C}^{n \times n} \) is a linear function of \( x \). Consider:

\[
\begin{align*}
\min_{x,\lambda} \quad & \lambda \\
\text{s.t.} \quad & \lambda \text{ is the largest eig. of } A(x)
\end{align*}
\]

Eigenvalue definition (informal):

\[
A(x)v = \lambda v \Rightarrow v^*A(x)v = \lambda v^*v
\]

\[
\Rightarrow v^*A(x)v = \lambda
\]

\[
\Rightarrow \max_{v \in \mathbb{C}^n} \frac{v^*A(x)v}{v^*v} = \lambda_{\max}
\]

...Rayleigh quotient. This implies:

\[
\lambda_{\max} v^*Iv \geq v^*A(x)v \quad \forall v \in \mathbb{C}^n
\]

Equivalent to

\[
\begin{align*}
\min_{\lambda,x} \quad & \lambda \\
\text{s.t.} \quad & v^*(\lambda I - A(x))v \geq 0 \quad \forall v \in \mathbb{C}^n
\end{align*}
\]

which, by definition of PSD, is equivalent to

\[
\begin{align*}
\min_{\lambda,x} \quad & \lambda \\
\text{s.t.} \quad & \lambda I - A(x) \succeq 0
\end{align*}
\]
3.4 Quadratically constrained programming

\[ \min_x x^* C x \]
\[ \text{s.t. } x^* A_i x \leq b_i \]

How difficult?
- If \( C \succeq 0 \) and \( A_i \succeq 0 \), solvable in PT.
- If any are not PSD, NP-hard.

How general?
- Binary constraints: \( x \in \{0, 1\} \iff x^2 = x \)
- Power flow: \( v_1 (v_1 - v_2)^* y_{12}^* = ... \)
- Both nonconvex!

3.5 Relaxations

Hard problem:
\[ F_1 : \min_{x \in X} f(x) \]

Relaxation:
\[ F_2 : \min_{x \in Y} f(x), \quad X \subset Y \]

Facts
- Obj. of \( F_2 \leq \) Obj. of \( F_1 \).
- If \( x \) is optimal for relaxation and feasible for exact, \( x \) is optimal for exact.

**Proof:** Suppose \( x \) is relaxed optimal and feasible suboptimal for exact problem. Then \( \exists y \) s.t. \( f(y) < f(x) \), \( y \in X \). But by relaxation, \( y \in Y \), and therefore \( x \) is not relaxed optimal, a contradiction. QED.

3.5.1 SD relaxation

Trace is invariant under cyclic permutations. QCP can be equivalently written:

\[ \min_x \text{trace}(xx^* C) \]
\[ \text{s.t. } \text{trace}(xx^* A_i) \leq b_i \]
Identical to:
\[
\begin{align*}
\min_{x,X} & \quad \text{trace}(XC) \\
\text{s.t.} & \quad \text{trace}(XA_i) \leq b_i \\
& \quad X = xx^* \iff X \succeq 0, \ \text{rank}(X) = 1
\end{align*}
\]

\(X = xx^*\) by itself is equivalent to \(X \succeq 0\) (unique Cholesky decomposition), \(\text{rank}(X) = 1\).

Removing a constraint enlarges the feasible set, i.e. relaxation:
\[
\begin{align*}
\min_X & \quad \text{trace}(XC) \\
\text{s.t.} & \quad \text{trace}(XA_i) \leq b_i \\
& \quad X \succeq 0
\end{align*}
\]

If solution, \(X\), has rank 1, then relaxation is tight. Feasible, optimal exact solution is Cholesky: \(X = xx^*\).

3.5.2 Example: Max-cut

Adjacency matrix
\[
A_{ij} = \begin{cases} 
1 & \text{if } i \sim j \\
0 & \text{otherwise}
\end{cases}
\]

Find the biggest cut (draw)... NP-complete. Mathematically:
\[
\begin{align*}
\max_x & \quad \frac{1}{2} \sum_{ij} A_{ij}(1 - x_i x_j) \\
\text{s.t.} & \quad x_i \in \{-1, 1\}
\end{align*}
\]

Equivalence:
\(x_i \in \{-1, 1\} \iff x_i^2 = 1\).

Convex relaxation: \(X = xx^T\) Equivalent formulation:
\[
\begin{align*}
\max_x & \quad \frac{1}{2} \sum_{ij} A_{ij}(1 - X_{ij}) \\
\text{s.t.} & \quad X_{ii} = 1, X \succeq 0
\end{align*}
\]